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Consistency Conditions for Fundamentally Discrete Theories

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Abstract

The dynamics of physical theories is usually described by differential equations. Difference equations then appear mainly as an approximation which can be used for a numerical analysis. As such, they have to fulfill certain conditions to ensure that the numerical solutions can reliably be used as approximations to solutions of the differential equation. There are, however, also systems where a difference equation is deemed to be fundamental, mainly in the context of quantum gravity. Since difference equations in general are harder to solve analytically than differential equations, it can be helpful to introduce an approximating differential equation as a continuum approximation. In this paper implications of this change in view point are analyzed to derive the conditions that the difference equation should satisfy. The difference equation in such a situation cannot be chosen freely but must be derived from a fundamental theory. Thus, the conditions for a discrete formulation can be translated into conditions for acceptable quantizations. In the main example, loop quantum cosmology, we show that the conditions are restrictive and serve as a selection criterion among possible quantization choices.

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I. INTRODUCTION

While differential equations play an outstanding role throughout physics, it has long been expected that a quantization of gravitational systems will lead to a discrete structure of space-time. Then, the differential equations must be replaced by difference equations at a fundamental level. The process of discretization itself is, in fact, well-known since it is used to study differential equations numerically. In this case the difference equation is only an approximation to the differential equation which is better suited to a numerical analysis. When the fundamental equation is discrete, however, the point of view must change. Now, analytic calculations are more complicated and less exact techniques are available than would be for a differential equation. Thus, it becomes convenient to consider, in certain regimes, a differential equation as an approximation to the difference equation. This is also necessary since the continuum formulation gives an excellent description of the real world such that there must be a continuum approximation to the discrete theory.

In both cases the relation between a difference and a differential equation is analyzed. But there is an important difference: In numerics [1] one has a large freedom in choosing the difference equation which can be exploited to obtain a very good approximation to the differential equation with helpful properties for the numerical analysis. If the difference equation is fundamental, e.g. if it follows from a basic quantum theory, then it cannot be changed so freely. Only quantization choices and ambiguities can be exploited which are often more limited and have to respect other consistency conditions of the theory. In particular, while the order of the difference equation should not be less than that of the differential equation to allow the same freedom as in the continuum formulation, there is no guarantee that it is not of considerably higher order. Also in numerics higher order schemes are being used to improve the numerical results, but one has to choose the higher order discretization very carefully. Whenever one uses a higher order, there are additional solutions, unwanted from the numerical point of view, which have to remain suppressed throughout the evolution. When the fundamental difference equation is of higher order than a continuum differential equation, there will also be additional solutions which correspond to a new freedom of the theory not seen in the continuum picture. While those solutions can play a physical role in certain regimes, their role must be limited in semiclassical regimes for the continuum formulation to be valid. Now, one has to check if the freedom allowed

by different quantization choices can not only provide a discrete equation which has the expected continuum approximation, but also, in the case of higher order, does not lead to growing modes which would soon dominate the continuum approximations.

The aim of this paper is to extract conditions for a fundamental difference equation to have a good continuum picture in this sense. Moreover, we will discuss how these conditions can be used as selection criteria among different quantizations or ambiguities. It turns out that they can be quite restrictive, thus having implications for quantization issues which at first glance seem only indirectly related to the fundamental law.

Our motivation, and the main example for a theory where in fact a discrete fundamental evolution law has been derived, is loop quantum cosmology. The issue of discretization itself as a consequence of quantum theory or gravity has been studied before [2], but in particular the issue of higher order has not always been appreciated. In order to cover also other examples of a discrete formulation, we will hold the discussion more general, but will use the example of loop quantum cosmology as an illustration throughout the paper. Moreover, we expect that loop quantum cosmology, or more generally quantum geometry which it is a part of, will be the prime application of the selection criteria. The reason is that quantum geometry is the most advanced formulation which has led to a discrete structure. Its kinematical setup is at a mathematically precise and well-defined level which demonstrates the main quantization ambiguities involved in the definition of the basic laws, mainly the Hamiltonian constraint. In such a scenario the selection criteria will be most useful since they can be tested explicitly and can restrict the possible choices. While explicit calculations in the full theory are still outside current reach, one can study the basic law explicitly in homogeneous models. Most of them, those with non-zero intrinsic curvature, require an additional input compared to the full theory such that the contact is not so close and more ambiguities arise [3]. It is then helpful to use the criteria discussed here in order to select a quantization.

In section II we specify the class of difference equations considered. We briefly recall the quantization of homogeneous models and describe the form of the ensuing fundamental difference equations.

In section III, we discuss the intuitive notion of continuum approximation and its formulation in terms of pre-classical solutions. For definiteness and clarity, in this section we discuss fundamental equations which are ordinary difference equations. The notion of pre-

classicality is carefully defined without appealing to any limiting procedure (Def. 2). Its implications for the difference equation are detailed. Subsequently a constructive procedure is given to demonstrate the existence of pre-classical solutions. This procedure naturally leads to a notion of local stability which is defined here (Def. 5). The conditions of admissibility of a pre-classical solution together with local stability are given in terms of the coefficients of the fundamental difference equation. These are our main results summarized in Props. 1 and 2.

In section IV, the case of a fundamental partial difference equation is analyzed. The analysis closely follows the steps detailed in section III. In some cases the fundamental equation can be interpreted as a non-homogeneous ordinary difference equation and this can be done in several ways. For each of these, the definition of local stability from section III can be applied. The fundamental difference equation can then be constrained by requiring that the local stability criteria must be satisfied for any evolutionary interpretation. The example of homogeneous loop quantum cosmology [3] is described briefly.

In the concluding section V, we summarize our results.

An appendix is included which illustrates an example of an older, unstable quantization of isotropic loop quantum cosmology.

II. DIFFERENCE EQUATIONS

In this section we fix our general notation of the class of difference equations considered, and describe the more special way in which they arise from loop quantum cosmology.

A. General set-up

We assume that there is a continuum formulation for the system under consideration which has a number of continuous parameters p^I (which can be space or space-time coordinates, or in the case of cosmology minisuperspace coordinates) and a linear, homogeneous differential equation $\mathcal{D}s = 0$ of order κ for a (wave) function $s(p^I)$. These basic variables have an underlying discrete structure given by the lattice

$$p^I(m_I) = \gamma \ell m_I \tag{1}$$

where m_I are integers and γ is the dimensionless parameter which controls the scale of the discreteness. The parameter ℓ will always be fixed and can absorb possible dimensions. At the discrete level, the fundamental equation is a difference equation of the form

$$\sum_{\{i_I=-k\}}^{k} A^{\{m_I\}}_{\{i_I\}}(\gamma)\; s_{\{m_I+i_I\}} = 0 \tag{2}$$

of order $2k$ for a (wave) function $s_{\{m_I\}}$ living on the lattice (the notation $\{\ldots\}$ in the sum indicates that all i_I are summed over). The coefficients can in general depend on the coordinates m_I as well as the parameter γ.

Due to the relation between p^I and m_I we expect a relation between the discrete and the continuous formulation when γ is small and the m_I are large, which will be assumed from now on. If γ is small but non-zero, the differential equation can only be an approximation to the difference equation; a strict relation could be achieved in the limit $\gamma \to 0$, $m_I \to \infty$ such that p^I is unchanged, which we will call, as in loop quantum cosmology, the pre-classical limit. Since a relation between the formulations will be demonstrated by using expansions

$$s_{\{m_I+i_I\}} = s(p^I(m_I)) + \sum_I \gamma i_I \partial_I s(p^I(m_I)) + \cdots \tag{3}$$

one also has to assume properties of the wave function concerning its oscillation on small scales. This will be discussed in more detail later.

We will also assume that the coefficients of the difference equation can be expanded in terms of γ,

$$A^{\{m_I\}}_{\{i_I\}}(\gamma) = \sum_{j=-l}^{\infty} \gamma^j A^{(j)}_{\{i_I\}}(p^I(m_I)) \tag{4}$$

where poles at most of order l appear. (Some, but not all of the $A^{(-l)}_{\{i_I\}}(p^I)$ can be zero.) Note that we use the p^I to parameterize the coefficients in the expansion. This is necessary because later we will use the expansion in the continuum limit where some of the γ-dependence is absorbed in the m_I. Furthermore, we assume the following conditions for the coefficients to be satisfied at least for large m_I which would be used in a pre-classical limit or approximation (at small m_I, the conditions can be violated by quantum effects):

$$\text{unitarity conditions} \quad A^{\{m_I\}}_{\{i_I\}} = \left(A^{\{m_I\}}_{\{-i_I\}}\right)^* \tag{5}$$

$$\text{parity conditions} \quad A^{\{m_I\}}_{\{i_I\}} = \pm A^{\{-m_I\}}_{\{i_I\}}\,. \tag{6}$$

The first condition ensures that the equation with a reversal of the direction of "time" is solved by the complex conjugate of the original wave function:

$$\sum_{\{i_I=-k\}}^{k} A^{\{m_I\}}_{\{i_I\}} s^*_{\{m_I-i_I\}} = \sum_{\{i_I=-k\}}^{k} A^{\{m_I\}}_{\{-i_I\}} s^*_{\{m_I+i_I\}} = \left(\sum_{\{i_I=-k\}}^{k} A^{\{m_I\}}_{\{i_I\}} s_{\{m_I+i_I\}} \right)^* = 0$$

while the second one guarantees that a solution remains a solution when all the coordinates are reversed.

Depending on the situation and the physical meaning of the parameters p^I, we will usually assume rotational symmetry of the equations, i.e. permutation symmetry for the coefficients m_I.

B. Loop quantum cosmology

In diagonal homogeneous loop quantum cosmology [4] we have three independent variables p^I, $I = 1, \ldots, 3$, which are the components of the densitized triad. They determine the spatial metric as $\mathrm{d}s^2 = a_I^2 (\omega^I)^2$ where ω^I are fixed invariant 1-forms which specify the particular model, and $a_I = \sqrt{|p^J p^K / p^I|}$ if $\epsilon_{IJK} = 1$. If we restrict to isotropic models, there is a single p and $a = \sqrt{|p|}$ is the scale factor. The discreteness is controlled by the Barbero–Immirzi parameter γ of quantum geometry, and $\ell = \frac{1}{2}\ell_{\mathrm{P}}^2$ with the Planck length ℓ_{P} (or $\ell = \frac{1}{6}\ell_{\mathrm{P}}^2$ in the isotropic case). Note that the regular lattice discretization $p^I(m_I) = \frac{1}{2}\gamma\ell_{\mathrm{P}}^2 m_I$ is not an assumption but derived from the theory via the spectra of densitized triad operators; it is a general feature of quantum geometry. For diagonal homogeneous models the lattice structure follows from the eigenvalues $p^I(m_I)$ of triad operators $\hat{p}^I$ with eigenstates $|m_1, m_2, m_3\rangle$. A wave function is supported on this lattice, s_{m_1,m_2,m_3} in the triad representation defined by $|s\rangle = \sum_{\{m_I\}} s_{m_1,m_2,m_3} |m_1, m_2, m_3\rangle$ for an arbitrary state $|s\rangle$. Since the states $|m_1, m_2, m_3\rangle$, defined as eigenstates of the triad operators, have the freedom of an arbitrary (possibly m_I-dependent) phase factor, the triad representation and the fundamental equation for a wave function are fixed only after a choice of the phase factors has been made.

Another essential part of quantum geometry is that the densitized triad is conjugate to a connection, which in the diagonal homogeneous case also has three components $c_I = \Gamma_I - \gamma K_I$ where Γ_I is the spin connection and K_I the extrinsic curvature. It is quantized in quantum geometry not directly, but via its holonomies [5]. In the homogeneous case, these

holonomies are given essentially by $\sin(\frac{1}{2}c_I)$ and $\cos(\frac{1}{2}c_I)$ acting as multiplication operators in the connection representation. If we choose the phases for the triad eigenvectors as

$$\langle c_1, c_2, c_3 | m_1, m_2, m_3\rangle = 2^{-\frac{3}{2}} \frac{\exp(i(m_1c_1 + m_2c_2 + m_3c_3)/2)}{\sin(c_1/2)\sin(c_2/2)\sin(c_3/2)} \tag{7}$$

the holonomy operators act on a wave function $s_{\{m_I\}}$ via, e.g.,

$$(\sin(\tfrac{1}{2}c_1)s)_{m_1,m_2,m_3} = \tfrac{1}{2}i(s_{m_1+1,m_2,m_3} - s_{m_1-1,m_2,m_3}) \tag{8}$$

$$(\cos(\tfrac{1}{2}c_1)s)_{m_1,m_2,m_3} = \tfrac{1}{2}(s_{m_1+1,m_2,m_3} + s_{m_1-1,m_2,m_3})\,. \tag{9}$$

The fundamental equation is given by the Hamiltonian constraint equation for a wave function. In the full theory, a part of it is constructed with a Wilson loop evaluated in the connection. It can straightforwardly be specialized to a homogeneous model provided that there is no intrinsic curvature (i.e., the Bianchi I model). To respect the symmetry we are only allowed to use integral curves of vector fields generating the symmetry (dual to the invariant 1-forms used above) in forming the Wilson loop; this can be done by forming a square from four such curves which close because the vector fields commute. The Hamiltonian constraint equation for the Bianchi I model is then essentially fixed taking the form (13) with non-zero coefficients

$$A^{m_1,m_2,m_3}_{2\epsilon_1,2\epsilon_2,0}(\gamma) = \epsilon_1\epsilon_2\,\gamma^{-3}\kappa^{-1}\ell_{\rm P}^{-2}(V_{m_1-2\epsilon_1,m_2-2\epsilon_2,m_3+1} - V_{m_1-2\epsilon_1,m_2-2\epsilon_2,m_3-1}) \tag{10}$$

for $\epsilon_I = \pm 1$, and similarly for other coefficients by permutation symmetry. We used the volume eigenvalues

$$V_{m_1,m_2,m_3} = (\tfrac{1}{2}\gamma\ell_{\rm P}^2)^{\frac{3}{2}}\sqrt{|m_1m_2m_3|} \tag{11}$$

to simplify the notation, $\ell_{\rm P}^2 = \kappa\hbar$, and $\kappa = 8\pi G$ is the gravitational constant. If there is matter or a cosmological constant, there will also be a coefficient (it can be operator valued, acting on the matter dependence of the wave function)

$$A^{m_1,m_2,m_3}_{0,0,0}(\gamma) = \hat{H}_{\rm matter}(m_1,m_2,m_3)\,. \tag{12}$$

In the notation of (4) we have a pole of order $l = 2$ and non-vanishing coefficients which can be read off from

$$\begin{aligned} V_{m_1-2\epsilon_1,m_2-2\epsilon_2,m_3+1} &- V_{m_1-2\epsilon_1,m_2-2\epsilon_2,m_3-1} \\ &= \sqrt{|p^1-\epsilon_1\gamma\ell_{\rm P}^2||p^2-\epsilon_2\gamma\ell_{\rm P}^2|}\left(\sqrt{p^3+\tfrac{1}{2}\gamma\ell_{\rm P}^2}-\sqrt{p^3-\tfrac{1}{2}\gamma\ell_{\rm P}^2}\right) \\ &= \tfrac{1}{2}\gamma\ell_{\rm P}^2\sqrt{|p^1p^2/p^3|}\left(1-\tfrac{1}{2}\gamma\ell_{\rm P}^2(\epsilon_1/p^1+\epsilon_2/p^2)+O(\gamma^2)\right)\,. \end{aligned}$$

The order of the partial difference equation is $2k = 4$.

The fact that we have a difference equation can be traced back to the Wilson loop operator which can be decomposed into a product of the operators (8), (9) giving us the basic difference and mean. It also demonstrates the point that a fundamental difference equation can easily be of higher order: the basic difference (8) is a second order difference operator, rather than first order (corresponding to a leap-frog scheme in numerics). It is a basic result of the theory and cannot be changed to simplify the equations.

In other Bianchi models with non-zero intrinsic curvature the situation is not so clear-cut. Their symmetries are generated by vector fields which do not commute, thus not forming closed loops in the way described above. We have to correct for this fact by making another choice which *cannot* be motivated from the full theory. There needs to be an additional step because non-zero intrinsic curvature in a model is realized by a non-zero spin connection. While this is an invariant statement in the model, it does not make sense in the full theory where the spin connection can be made to vanish locally. Therefore, it does not pose a problem in the full theory, but we have to face it in any model where it does not vanish [3]. Guidance by the full theory, which we could use for the Bianchi I model, is lost which results in more quantization choices and ambiguities.

The purpose of the conditions studied in this paper is two-fold. We have to check if they are fulfilled in the Bianchi I model which then, due to its close relation to the full theory, can be regarded as a consistency check for the full quantization. For the other models we can use the conditions to select among the possibilities for a quantization (if they are possible to realize at all).

C. Pre-classical approximation vs. pre-classical limit

It is obvious from the basic definitions that the smaller the parameter γ, the closer we are to a continuum formulation. This also requires to have large values of the discrete variables m_I, and only small variations of the wave function $s_{\{m_I\}}$ between adjacent lattice points. If we perform the continuum limit $\gamma \to 0$ exactly, then we must simultaneously have $m_I \to \infty$ and s must become a smooth function $s(p^I)$; this is the *pre-classical limit* [6]. From a different conceptual point of view, we can also work with small but non-zero values of γ, which is the physically correct way in the case of loop quantum cosmology

where $\gamma = \log(2)/\pi\sqrt{3}$ can be inferred from calculations of black hole entropy in quantum geometry [7, 8]. In this case we still have to consider large m_I and small oscillations of the wave functions at Planck scales for the continuum *pre-classical approximation* to be valid. This corresponds to a regime of large volume and small curvature in the case of gravity.

Concerning the main theme of this paper, the first point of view of a pre-classical limit is the one usually taken in numerics. When the differential equation is regarded as fundamental and the difference equation as an approximation, the limit has to be performed eventually such that γ can be assumed to be as small as necessary for a particular mathematical result. Thus, mathematical techniques developed for the numerical analysis of differential equations can be used when studying the pre-classical limit. When the difference equation is fundamental, however, γ has a physical value which is non-zero. In such a situation results which rely on the fact that γ can be chosen as small as appropriate, will not apply and new definitions and theorems have to be developed. In this paper we present a first approach giving basic definitions and some implications.

III. CONDITIONS ON THE FUNDAMENTAL DIFFERENCE EQUATION

To simplify the presentation let us first work in the context of isotropic loop quantum cosmology [9] so that we have a single triad component $\hat{p}$ with eigenvalues $p(m) = \frac{1}{6}\gamma\ell_{\mathrm{P}}^2 m$ and eigenstates $|m\rangle$. The fundamental difference equation is of the general form (2) with only one summation label,

$$A_k^m(\gamma)s_{m+k} + A_{k-1}^m(\gamma)s_{m+k-1} + \cdots + A_{-k}^m(\gamma)s_{m-k} = 0\,, \tag{13}$$

where the coefficients satisfy the unitarity conditions for large m,

$$A_{-i}^m \simeq (A_i^m)^*\,. \tag{14}$$

The γ-expansion of the coefficients is such that only one pole term of order $l = 2$ appears, while all other terms have non-negative power of γ.

Note: In practice, the difference equation resulting from a quantization of the Hamiltonian constraint is not unique. As discussed in the preceding section, the discrete wave functions s_m are defined only modulo m-dependent phase transformations which also alters the coefficients of the difference equation. Such a freedom has to be kept in mind while checking the

conditions we will derive for any particular difference equation. For *homogeneous* difference equations which arise in the isotropic context, the coefficients are defined only up to common factors. Additionally, for large m in particular, such common factors do occur but can be canceled. In the following, the coefficients will be assumed to have no such common factors.

A. Continuum Approximation

When a discrete formulation is used at the fundamental level, it has to reproduce the familiar continuum idealization at large scales or in semiclassical regimes as a very good approximation. This presents a necessary condition for any discrete formulation; in particular a quantization of geometry has to reproduce the familiar continuum metric formulation as an approximation when quantum effects can be expected to be negligible. The latter is realized when the length scale on which the continuum formulation is used is large compared to the Planck scale on which quantum geometry is to be used, and curvature is not significant. In the context of homogeneous cosmology the corresponding region of phase space has large values of the triads (and by implication of volume) and small values of the extrinsic curvature. The intrinsic curvature of a minisuperspace model, by contrast, is a fixed function on the phase space and thus cannot be independently required to be small in a semiclassical regime; it can be large even in regions where the volume is large and the extrinsic curvature is small, which has to be taken care of properly.

For gravity, the familiar Wheeler–DeWitt quantization of homogeneous models [10, 11] is obtained using a Schrödinger quantization of the continuous geometry variables, i.e. the spatial metric and the extrinsic curvature. In this quantization, the fundamental formulation is still continuous: wave functions are functions of continuous labels p and the constraint equation leads to the Wheeler–DeWitt *differential* equation. Since the Wheeler–DeWitt quantization already uses continuous geometric variables, it is convenient (but not necessary since, after all, the Wheeler–DeWitt quantization is not well-defined for the full theory) to require that the wave functions of this quantization together with the Wheeler–DeWitt equation they satisfy should be reproduced as a suitable *limit* or at least as the leading behavior in an *approximation* from some solution(s) of the fundamental difference equation.

The indication of the regime of semiclassical behavior and the justification for matching with the Wheeler–DeWitt quantization can be further seen by observing that in a triad

representation, the basic difference operators of the discrete formulation are given by $\sin(\frac{1}{2}c)$ as in (8), while the basic differential operators of a Wheeler–DeWitt quantization are given by $\hat{c} = i\gamma\ell_{\rm P}^2\partial/\partial p$. Thus, if c can be assumed to be small, we in fact have $\sin(\frac{1}{2}c) \sim \frac{1}{2}\hat{c}$ and the difference operators immediately yield differential operators for small c. This assumption, however, is justified only in the simplest cosmological models, the Bianchi I model and its isotropic sub-models in which the intrinsic curvature, represented by the spin connection Γ, is zero and c is essentially the extrinsic curvature. Otherwise, c also has a contribution from intrinsic curvature and the relation to the continuum formulation is less direct; its proper treatment is discussed in [3, 12].

The condition of large volume is easier to implement since the large volume regime naturally corresponds to large eigenvalues of the triad operator, $m \gg 1$. If finite increments of the label m appearing in the difference equation have only small effects on the wave function (which also depends on the curvature) then one can expect difference operators acting on sufficiently slowly varying functions of m to be well approximated by differential operators. In this way, one can conceive the possibility of recovering the Wheeler–DeWitt quantization. Thus, the basic idea is to see if there are 'sufficiently slowly varying' discrete wave functions, s_m which can be interpolated by 'sufficiently slowly varying' solutions of the Wheeler–DeWitt equation. The requirement that the Wheeler–DeWitt quantization be obtained as a good approximation translates into the requirement that the fundamental difference equation admit at least one such slowly varying solution. We will now make these statements precise and deduce their implications.

1. Preliminaries

In the following, the *domain of validity* of a continuum approximation is specified by $1 \ll M_{\rm min} \ll M_{\rm max}$ and $m \in (M_{\rm min}, M_{\rm max})$. For m_0 in the domain of validity, we will specify its Δm-neighborhood as the interval $[m_0, m_0 + \Delta m]$. Statements valid over Δm-neighborhoods will be termed as *local* statements. Typically, $1 \ll \Delta m \ll m_0$ will be assumed. In some situations the domain of validity can extend over an infinite range of m.

The coefficients in the difference equation depend on γ and m. The γ dependence is such that $A_i^m = \gamma^{-2}(A^{(-2)})_i^m + (A^{(0)})_i^m + \gamma(A^{(1)})_i^m + \cdots$. Note that there is a term of order γ^{-2} while there is no γ^{-1} term in the coefficients, as discussed in Section II B. All the $(A^{(r)})_i^m$

also depend on m which, in a Δm neighborhood of some m_0 in the domain of validity, can be expanded as: $(A^{(r)})_i^{m_0+\delta m} \simeq A_i^{(r)} + \frac{\delta m}{m_0} B_i^{(r)} + \cdots$, $0 \leq \delta m \leq \Delta m$, where $A_i^{(r)}, B_i^{(r)}$ are constants depending on the fixed m_0 but independent of δm.

A few preliminary definitions: A function ψ of a continuous variable p is said to be *locally slowly varying around* p_0 if $\psi(p)$ can be Taylor expanded about p_0 with succeeding terms smaller than the preceding terms for $p \in [p_0, p_0 + \Delta p]$. A bound on the ratios can be used to quantify degree of slow variation but we will not need it. We will typically need terms only up to second order. Correspondingly, a sequence s_m is said to be *locally slowly varying around* m_0 if it can be obtained via $s_m := \psi(p(m))$ where $p(m) = \frac{1}{6}\gamma\ell_{\mathrm{P}}^2 m$, $\psi(p)$ is locally slowly varying. In other words, a sequence is slowly varying if it can be interpolated by a slowly varying function, which, of course, will not be unique.

Notice that the definition of the variable $p(m)$ contains an explicit γ dependence. Thus, even if a slowly varying $\psi(p)$ has no γ dependence, the corresponding slowly varying sequence does depend on γ and is in fact *Taylor expandable in* γ. Conversely, if a slowly varying sequence has a specific type of γ dependence (because it is a solution of the difference equation containing γ, for instance), the corresponding interpolating function can be independent of γ.

Thus, a slowly varying function and a slowly varying sequence can be written as

$$\psi(p_0 + \delta p) \;\simeq\; \psi(p_0) + \delta p\; \psi'(p_0) + \frac{1}{2}\delta p^2 \psi''(p_0) + \cdots \;, \tag{15}$$

$$s_{m_0+\delta m} \;\simeq\; \psi(p(m_0)) + (\gamma\ell_{\mathrm{P}}^2/6)\,\delta m\; \psi'(p(m_0)) + \frac{1}{2}(\gamma\ell_{\mathrm{P}}^2/6)^2\,\delta m^2\,\psi''(p(m_0)) + \cdots \tag{16}$$

Let us assume now that the fundamental equation admits solutions which are locally slowly varying around some m_0 in the domain of validity.

2. Pre-classicality

Using the γ-expansions of the coefficients and substituting (16) for a locally slowly varying s_{m+i} in the fundamental difference equation (13) leads to,

$$\sum_{i=-k}^{k} \left(\gamma^{-2}(A^{(-2)})_i^m + (A^{(0)})_i^m + \gamma(A^{(1)})_i^m + \cdots\right) \times \left(\psi(p(m)) + (\gamma\ell_{\mathrm{P}}^2/6)\, i\, \frac{\mathrm{d}\psi}{\mathrm{d}p} + \frac{1}{2}(\gamma\ell_{\mathrm{P}}^2/6)^2\, i^2\, \frac{\mathrm{d}^2\psi}{\mathrm{d}p^2} + \cdots\right) \;=\; 0 \tag{17}$$

It is clear that we can get a *differential equation* for $\psi(p)$ in this manner for *every locally slowly varying solution* s_m of the fundamental equation.

Essential for a continuum approximation is not just to get a differential equation but that a differential equation so obtained should be *independent of* γ. The γ independence condition is necessary since by definition γ controls the discreteness such that the continuum variables are obtained for $\gamma \to 0$. For instance, in the case of gravity the differential equation is expected to be the standard Wheeler–DeWitt equation which knows nothing about γ.

It follows immediately that $\psi(p)$ must satisfy equation(s) obtained by equating coefficients of powers of γ to zero. There are, of course, infinitely many such equations.

From any of these equations, one can derive a 'Hamilton–Jacobi' equation for the phase of the wave function in the usual manner and read off *a classical Hamiltonian constraint*. Only one of these equations reproduces the correct classical Hamiltonian constraint. In view of the small value of γ, this equation satisfied by $\psi(p)$ is to be considered as the *leading approximation* and consequently, the equations corresponding to lower powers of γ must be identically zero *without* any conditions on $\psi(p)$.

Thus we define:

Definition 1 *A locally slowly varying solution around m_0 of the fundamental difference equation is said to be* locally pre-classical around m_0 *if there is an interpolating slowly varying function $\psi(p)$ which solves a differential equation that has no dependence on γ and has the correct classical limit.*

Thus, the γ independence condition together with the correct *classical limit*, implies that the $o(\gamma^{-2}), o(\gamma^{-1})$ terms must be zero identically and the $o(\gamma^0)$ term must contain derivatives of $\psi(p)$. Furthermore, in a Δm-neighborhood of m_0 we can use $(A^{(r)})^m_i \simeq A^{(r)}_i$ which also removes the explicit m dependence. This leads to:

$$\sum_{i=-k}^{k} A_i^{(-2)} = 0 = \sum_{i=-k}^{k} i A_i^{(-2)} \ , \quad \text{and} \quad \sum_{i=-k}^{k} A_i^{(-2)}\ i^2 \neq 0 \qquad \forall\ m \in (m_0, m_0 + \Delta m) \ , \tag{18}$$

and we obtain, to the leading order in γ and $\frac{\Delta m}{m_0}$, the *approximating differential equation* as

$$\frac{1}{72}\ell_{\rm P}^4 \left(\sum_{i=-k}^{k} A_i^{(-2)}\ i^2\right) \frac{d^2\psi}{dp^2} + \left(\sum_{i=-k}^{k} A_i^{(0)}\right) \psi(p) + o(\gamma, \Delta m/m_0) = 0 \ . \tag{19}$$

The approximating differential equation will precisely match with the Wheeler–DeWitt equation if the ratio of the coefficients of the two terms on the left hand side of (19) matches with those of the Wheeler–DeWitt equation. This indeed turns out to be the case. We will refer to the leading order approximating differential equation as the Wheeler–DeWitt equation. Using the Wheeler–DeWitt equation for capturing the continuum behavior is thus an approximation. From the perspective of the discrete solutions, locally pre-classical solutions are still *exact* solutions of the fundamental equation.

Thus we have seen that the admissibility of even a *single* locally pre-classical solution puts strong restrictions on the coefficients of the fundamental equation:

Proposition 1 *If a difference equation of the form (13) with coefficients having a second order pole in the parameter γ admits a locally pre-classical solution, then the leading order terms of the coefficients, $A_i^{(-2)}$, must satisfy (18).*

We are now ready to define pre-classical solutions.

Definition 2 (Pre-Classicality) *A solution of the fundamental equation is said to be* pre-classical *if it is locally pre-classical around every m_0 in the domain of validity.*

The requirement of the existence of a pre-classical solution implies that all the local statements above, in particular the conditions in (18) must hold throughout the domain of validity.

Notice that these conditions are homogeneous in the coefficients. For the equations of the isotropic homogeneous models, there are only *three* non-zero coefficients $A_i^{(-2)}$ and the two conditions of (18) fix these up to an over-all scaling. It is the over-all scaling that contains the m_0 dependence. The remaining ratios being independent of m_0 are the same for local neighborhoods around any m_0 in the domain of validity.

To summarize: We have reformulated the definition of a pre-classical solution *without* reference to any pre-classical limit [6]. It is more realistic since it works with small but non-zero γ and large but not infinite m, captures the essential ideas of getting a continuum description as an approximation in terms of solutions of the Wheeler–DeWitt equation and directly gives restrictive conditions (18) on the coefficients of the fundamental difference equation. One can construct *approximate* pre-classical solutions from (slowly varying) solutions of the Wheeler–DeWitt equation as outlined in [6].

B. Local Stability

We now address the issues of the existence of pre-classical solutions, the number of independent pre-classical solutions and the construction of approximate pre-classical solutions. We will work exclusively within the discrete formulation and assume that the coefficients of the fundamental equation satisfy the conditions of (18) which are necessary conditions for the existence of a pre-classical solution. As a consequence, we will construct local approximations and use them to argue about the existence of pre-classical solutions.

1. Approximate pre-classicality

Consider locally slowly varying solutions (pre-classical ones being a special case) of the fundamental equation. We know that such solutions are Taylor expandable in γ : $s_m = s_m^{(0)} + \gamma\ s_m^{(1)} + \gamma^2\ s_m^{(2)} + \cdots$. Substituting in (13) and using the fact that the coefficients are locally almost constant gives,

$$\begin{aligned} 0\ =\ \gamma^{-2} \left(\sum_{i=-k}^{k} A_i^{(-2)} s_{m+i}^{(0)} \right) + \gamma^{-1} \left(\sum_{i=-k}^{k} A_i^{(-2)} s_{m+i}^{(1)} \right) \\ +\gamma^0 \left(\sum_{i=-k}^{k} A_i^{(-2)} s_{m+i}^{(2)} + \sum_{i=-k}^{k} A_i^{(0)} s_{m+i}^{(0)} \right) + o(\gamma) \end{aligned} \tag{20}$$

It is evident now that a locally pre-classical solution can be constructed as a power series in γ by *choosing* $s_m^{(0)},\ s_m^{(1)}$ to be solutions of

$$\sum_{i=-k}^{k} A_i^{(-2)} s_{m+i} = 0\ , \tag{21}$$

while $s_m^{(2)}$ is a solution of the non-homogeneous equation

$$\sum_{i=-k}^{k} A_i^{(-2)} s_{m+i}^{(2)} + \sum_{i=-k}^{k} A_i^{(0)} s_{m+i}^{(0)} = 0\ . \tag{22}$$

The higher order terms are to be constructed recursively from solutions of *non-homogeneous* difference equations with constant coefficients. Truncating the Taylor series to any finite order constitutes an approximation. Thus we define:

Definition 3 *A solution of the form* $s_m = s_m^{(0)} + \gamma\ s_m^{(1)} + \gamma^2\ s_m^{(2)}$ *with* $s_m^{(r)}$ *satisfying (21,22) is said to be a 2nd order approximate, locally pre-classical solution or* 2nd order local approximation *for short.*

Apart from being Taylor expandable in γ, locally slowly varying sequences also have the form $s_{m+i} \simeq a_m + ib_m + i^2 c_m + \cdots$, where $b_m \sim o(\gamma),\ c_m \sim o(\gamma^2)$ etc. as seen in (16). In fact, the equations (21, 22) do admit solutions of this form. To see this let us take the ansatz (for $r = 0, 1, 2$) ,

$$s^{(r)}_{m_0+\delta m} \simeq a^{(r)}(m_0) + (m_0+\delta m) b^{(r)}(m_0) + (m_0+\delta m)^2 c^{(r)}(m_0) + \cdots \ , \ 0 \leq \delta m \leq \Delta m \ . \quad (23)$$

Substitution in the equations (21, 22) implies that $a^{(r)}(m_0),\ b^{(r)}(m_0)$ are arbitrary, $c^{(r)}(m_0) = 0$ for $r = 0, 1$ and $c^{(2)}(m_0)$ is determined in terms of the $a^{(0)}(m_0),\ b^{(0)}(m_0)$ and coefficients of the difference equations. The conditions (18) are crucial for these statements. We can *choose* the $a^{(r)}(m_0), b^{(r)}(m_0)$ coefficients so that $s_{m_0+\delta m}$ is obtained as a Taylor series in $\gamma\delta m$. Explicitly the choices are: $b^{(0)}(m_0) = 0,\ a^{(1)}(m_0) = -m_0 b^{(1)}(m_0),\ a^{(2)}(m_0) = -m_0^2 c^{(2)}(m_0),\ b^{(2)}(m_0) = -2m_0 c^{(2)}(m_0)$. This leaves us with *two* free parameters, $a^{(0)}(m_0),\ b^{(1)}(m_0)$. We have thus constructed 2nd order local approximations (in a Δm neighborhood of m_0 and to order γ^2) which are parameterized by two constants and given by,

$$s_{m_0+\delta m} = a^{(0)}(m_0) + \gamma\delta m b^{(1)}(m_0) + \gamma^2 \delta m^2 c^{(2)}(m_0) \ . \quad (24)$$

The pre-classicality properties of these solutions are obvious. It is also clear that the number of independent, exact pre-classical solutions can not be more than *two* since their local approximations are parameterized by two parameters[18] (this is true even if we go to higher orders in γ).

We can construct such two parameter families of approximate, locally pre-classical solution around various m_j in the domain of validity. These should somehow be tied together to constitute an approximation to a single (common) exact pre-classical solution, as discussed in what follows.

2. Construction of approximate, pre-classical solutions

Now we give a procedure to construct a solution which will approximate a pre-classical solution in various local neighborhoods. To do so, we divide the domain of validity into several, non-overlapping local neighborhoods around $m_0 := M_{\min}, m_1, \cdots, m_N := M_{\max}$ within each of which we have the constant coefficients $A_i^{(-2)}(m_j)$ satisfying the conditions

(18). We can construct local approximations in each of these local neighborhoods as outlined above. These local approximations now have to be 'matched' across the boundaries of the adjacent neighborhoods.

Both the exact (13) and the approximated (21, 22) evolution equations are of order $2k$, and to determine any particular solution we need $2k$ consecutive s_m's to be specified as *initial conditions*. We will generically refer to the set $\{s_m, s_{m+1}, \cdots, s_{m+2k-1}\}$ as *initial data at* m. In each neighborhood, the local approximations induce initial data of the form

$$s_{m_j+i} = a^{(0)}(m_j) + \gamma i b^{(1)}(m_j) + \gamma^2 i^2 c^{(2)}(m_j) + \cdots \ , \quad i = 0, 1, \cdots, 2k-1 \ . \tag{25}$$

Likewise, we also have the $2k$ consecutive values from $s_{m_j+\Delta m_j-2k}$ as,

$$s_{m_j+\Delta m_j-2k+i} = a^{(0)}(m_j) + \gamma(\Delta m_j - 2k + i) b^{(1)}(m_j) + \gamma^2(\Delta m_j - 2k + i)^2 c^{(2)}(m_j) + \cdots \ . \tag{26}$$

The initial data at $m_{j+1} = m_j + \Delta m_j$ however can not be arbitrary since the exact equation holds across the neighborhoods. We can still use the constant coefficients to connect these values, however the constant coefficients will now involve a mixture of coefficients from both the adjacent neighborhoods and this mixture need not satisfy the conditions (18). Equivalently, the initial data so constructed at $m_j + \Delta m_j$ need not equal the data induced from the local approximation in the neighborhood $[m_{j+1}, m_{j+1} + \Delta m_{j+1}]$. We will refer to the former as the *extended initial data*, the latter as the *local data* and the corresponding solutions of (21, 22) as *extended solution* and *local solution*. Note that the initial data consist of $2k$ values while we have a freedom of only *two* parameters in the local data. Since usually $2k > 2$, the two data sets are generically distinct and so are the two solutions. In particular, the extended solution *cannot* be a local continuum approximation. Using a local solution instead of an extended solution thus introduces errors which can accumulate with subsequent extensions to cover the full domain of validity. The error accumulation can even jeopardize the existence of a pre-classical solution itself. This is prevented by requiring 'stability' for the approximated evolution (21, 22).

To arrive at a suitable notion of stability for (21), let us concentrate on the equation:

$$\sum_{i=-k}^{k} A_i s_{m+i} \ = \ 0 \quad , \quad \forall \ m \in [m_j + k, m_j + \Delta m - k] \tag{27}$$

where A_i are constants which are assumed to satisfy the conditions (18). With the identification $A_i := A_i^{(-2)}$, this is the same as (21) and has already been referred to as the *locally approximated evolution*.

It is convenient to denote the initial data by a vector in $\mathbb{C}^{2k}$, $(\vec{S}(m_j))^i := s_{m_j+i}$, $i = 0, 1, \cdots, 2k-1$. On the space of initial data, we have the natural Euclidean norm $\mathbb{C}^{2k}, ||\vec{S}(m_j)||^2 := \sum_{i=0}^{2k-1} |s_{m_j+i}|^2$.

Consider now the approximated evolution of two 'nearby initial data'. Let $\vec{S}(m_j)$ be fixed initial data and consider other arbitrary initial data $\vec{S}'(m_j)$ such that $||\vec{S}'(m_j) - \vec{S}(m_j)|| =: \delta_{m_j} \leq \Delta$. These data will generate two solutions s'_m, s_m whose difference, δs_m, will also satisfy the approximated evolution equation. We would like to see how much the two solutions differ at $m_j + \Delta m - 2k$.

The general solutions of difference equations of constant coefficients are given by linear combinations of elementary solutions of the form $(z_a)^m m^{r_a}$ where z_a is a root of the *characteristic polynomial*, $\sum_i A_i z^i$, with multiplicity d_a and $0 \leq r_a \leq d_a - 1$ [14]. Notice that (18) implies that $z = 1$ is a root with multiplicity 2. Clearly, $\delta s_m = \sum_a \sum_{r_a=0}^{d_a} \delta C_{a,r_a} \, (z_a)^m m^{r_a}$ where a ranges over the distinct characteristic roots. Since $\vec{S}'(m_j)$ is arbitrary, consider a δs_m such that $\delta C_{a,r_a}$ is non-zero only for a particular elementary solution. Then we have,

$$\delta_{m_j+\Delta m-2k} \simeq |z_a|^{\Delta m-2k} \left(1 + r_a \frac{\Delta m - 2k}{m_j} + \cdots\right) \delta_{m_j} \ . \tag{28}$$

Clearly, if there is a characteristic root with absolute value greater than one, then the two solutions deviate exponentially. Taking the data $\vec{S}_{m_j}$ to be local data and $\vec{S}'_{m_j}$ to be the extended data, we see that even if the two data are within a Δ neighborhood initially, they could evolve far apart if $|z_a| > 1$. This leads us to the definition:

Definition 4 *The approximated evolution (27) is said to be* stable *if* $\delta_{m_j+i} \leq \Delta(1 + o(\frac{\Delta m_j}{m_j}))$ $\forall$ $0 \leq i \leq \Delta m - 2k$ *for all* $\delta_{m_j} \leq \Delta$.

A stable evolution then guarantees that the extended solution will be nearby the local solution i.e. $|\delta s_m| \leq \Delta$. It implies that *all* characteristic roots must have absolute value less than or equal to one. The coefficients of the difference equation also satisfy the complex conjugation property (14) which implies that if z is a root, so is $(z^*)^{-1}$. Combined with the stability requirement this implies

Proposition 2 *The approximated evolution of a difference equation (13) with coefficients satisfying the unitarity conditions (14) is stable if and only if all characteristic roots have absolute value one.*

As noted above, in the 'first neighborhood' ($[m_0, m_0 + \Delta m_0]$), we are left with a single free parameter, say $a^{(0)}(m_0)$, due to the consistency condition. In the next neighborhood we have two free parameters, $a^{(0)}(m_1), b^{(1)}(m_1)$. We can fix these by requiring that the extended and the local data at m_1 differ by the least amount eg. by a least square fit. This determines the free parameters and also determines δ_{m_1}. We can take $\Delta = \delta_{m_1}$. If the locally approximated evolution is stable in the second neighborhood, then the local solution in the second neighborhood is guaranteed to be within δ_{m_1} of the solution extended from the first neighborhood. If locally approximated evolutions are stable for all the neighborhoods, then we can continue our procedure and construct a sequence s_m with the property that it is an exact solution up to m_0 and is an approximate, locally pre-classical solution in every neighborhood of the chosen partition of the domain of validity. It is thus essential for the construction of an *approximate, piece-wise, pre-classical solution* that all the locally approximated evolutions be stable. This leads us to define:

Definition 5 (Local Stability) *The fundamental difference equation is said to be locally stable if the locally approximated evolution in any local neighborhood around any m_0 in the domain of validity, is stable. In particular this implies that the coefficients of the fundamental equation satisfying (18) have to satisfy the further condition that their corresponding characteristic polynomials have all roots with absolute value one.*

In other words,

$$\sum_{i=-k}^{k} A_i^{(-2)}(m_0) z^i(m_0) \;=\; 0 \;\text{ implies }\; |z(m_0)| = 1,\; \forall\; m_0 \;\in\; [M_{\min}, M_{\max}]\;. \qquad (29)$$

Note that the construction of an approximate pre-classical solution depends on the choice of $M_{\min}$ (up to which exact evolution is used) and also on the partitioning of the domain of validity. Hence such approximate pre-classical solutions are not unique. This is also true of *exact pre-classical solutions*. Since one does not know what initial data at $m = 0$ would characterize a pre-classical solution, we can choose the local data at any $m_0 \in [M_{\min}, M_{\max}]$ and evolve it exactly (in both directions) to generate *an* exact pre-classical solution. However the fact that approximate locally pre-classical solutions match with pre-classical solutions to order γ^2 and the property of local stability guarantees that a pre-classical solution defined from any local data and any of the approximate pre-classical solutions constructed, will always be 'nearby'. For instance, if we denote by ϵ the largest of the $\delta_j/\sqrt{2k}$ over all the

neighborhoods and all partitions, then we can expect any of the approximations or any of the exact pre-classical solutions to satisfy pairwise

$$|s_m - s'_m| \ < \ \epsilon \ \ \forall \ m \in [M_{\min}, M_{\max}] \tag{30}$$

Thus, our reformulated definition of pre-classicality, allows us to construct approximate pre-classical solutions to within ϵ-tolerance.

Several remarks are in order.

(1) The requirement of admissibility of a pre-classical solution and of local stability are both conceptually and logically independent. It is the requirement of admissibility of a pre-classical solution that implies the local constancy of the coefficients $(A^{(-2)})^m_i$ together with the condition (18) on these constants. These are explicitly picked out by the requirement of getting an approximating differential equation (Wheeler–DeWitt equation) which is independent of $\gamma,\ m$ explicitly. As a by-product, this requirement provides us with locally approximated evolutions and also approximate, locally pre-classical solutions which are parameterized by two free parameters. The local stability on the other hand is concerned with the construction of an approximate pre-classical solution by 'joining' together the various local approximations. Because the order of the fundamental evolution equation is always larger than two in loop quantum cosmology, the extension of local solutions introduces perturbations of local data which only stay small if the approximated evolution is stable. By contrast, a similarly locally approximated *differential* equation does not need stability for an extension since the initial data are local and we have exactly the same number of parameters as needed for matching.

(2) Although local stability is needed for approximate pre-classical solutions, the condition itself refers to the characteristic roots which are *independent of* the pre-classical solution. The stability property of the approximated evolution generically ensures that, compared to a given solution, nearby 'initial' data evolve into nearby 'final' data. If any of the non-pre-classical solution can also be approximated locally, then the extensions of its local approximations will involve similar perturbations of initial data. Note however that non-pre-classical solutions would generically be parameterized by more parameters $(2k-2)$ which are available for performing an exact matching and stability will not be a concern there (and not be necessary from a physical point of view).

(3) The implication of stability for the roots is independent of their multiplicities (less

than or equal to $2k-2$) which hardly affect the growth of deviations. The dangerous exponential growth is prevented by stability. Strictly speaking, the definition of stability allows roots to have absolute values to be $1+o((m_j)^{-1})$. With finite domains for local neighborhoods, one *cannot* hope to derive strict equalities. However, even without strict equalities, stability requirement puts severe constraints on the coefficients.

(4) It is important to note that the need for stability arises out of matching local approximations in the adjacent neighborhoods which always introduces differences between extended initial data and local initial data. It only matters that the local approximations are obtained as solutions of an initial value problem (i.e. evolution equations). Whether the evolution is via a homogeneous difference equation or via a non-homogeneous one, at best controls the difference between the two initial data sets. For the evolution of *differences* of solutions, it is *only* the homogeneous part of the evolution that matters. This observation will be particularly relevant for fundamental equations which are partial difference equations.

3. Further Implications for Loop Quantum Cosmology

Local stability provides a strict selection criterion which allows us to reduce the number of choices for a quantization of the Hamiltonian constraint in homogeneous models. This is because for a random polynomial with coefficients satisfying (14) it is very unlikely that all roots will have norm one. Also a quantization of the constraint using holonomy operators in some form will violate this condition in most cases. This is in particular helpful because models with non-zero intrinsic curvature require an additional input which is not present in the full theory and thus are less close to full quantum gravity. Only the Bianchi I model with vanishing intrinsic curvature does not require additional steps, and its quantization which has considerably less freedom turns out to be locally stable automatically.

Local stability was motivated by semiclassical considerations, and it has to be fulfilled only in semiclassical regimes. In the Planck regime it can be violated, and it usually is in the presence of matter or other effects which lead to large curvature. In such a situation it is expected that contributions to the wave function with Planck scale oscillations become dominant, leading to large differences between the discrete and a continuum formulation. Usually, an unbounded growth of the strongly oscillating solutions when one approaches

the classical singularity is suppressed by a consistency condition which follows from the constraint equation [6] as can, e.g., be seen from the discussion in [15] for the isotropic model.

To summarize our results for the case of isotropic loop quantum cosmology: Our requirements on the fundamental difference equation (13) are (a) the equation admits at least one pre-classical solution and (b) it is locally stable. The former leads to the conditions (18) while the latter requires that all the roots of the characteristic polynomial of the locally approximated difference equation with constant coefficients have unit absolute value. The conditions (18) also imply that among these roots is the root $\lambda = 1 + o(\gamma)$ with multiplicity two and this in turn is responsible for the two parameter family of approximate, locally pre-classical solutions. The fact that the multiplicity is two can directly be traced back to the second order pole in γ of the coefficients. A generalization to a different pole structure is straightforward.

IV. PARTIAL DIFFERENCE EQUATIONS

So far we concentrated on difference equations which were ordinary, homogeneous difference equations. Very often, one encounters *partial difference equations* as in the case of anisotropic, homogeneous loop quantum cosmology. The rationale and the logic for continuum approximations, pre-classicality etc remains the same but the details of course differ. We will concentrate on the differences one encounters and keep the discussion general. The concrete case of homogeneous loop quantum cosmology will be used as an example; it is discussed in more detail in [3].

A. Continuum Approximation

The fundamental partial difference equation is of the form,

$$\sum_{\{i_I\}} A^{\{m_I\}}_{\{i_I\}}(\gamma) s_{\{m_I+i_I\}} = 0 \quad \text{with} \quad A^{\{m_I\}}_{\{i_I\}}(\gamma) = \sum_{j=-l}^{\infty} \gamma^j (A^{(j)})^{\{m_I\}}_{\{i_I\}} \tag{31}$$

The locally slowly varying sequence is given in terms of an interpolating, locally slowly varying function $\psi(p)$ by (recall $p^I(m_I) = \gamma\ell m_I$),

$$s_{\{m_I+\delta m_I\}} \;=\; s_{\{m_I\}} + \gamma\ell \sum_J \delta m_J \partial_J \psi(p^I(m_I)) + \frac{1}{2}\gamma^2\ell^2 \sum_{J,K} \delta m_J \delta m_K \partial_J \partial_K \psi(p^I(m_I)) + \cdots \tag{32}$$

The γ-independent equations obtained from the fundamental equation become,

$$\left(\sum_{\{i_I\}}(A^{(-l)})^{\{m_I\}}_{\{i_I\}}\right)\psi(p^I(m_I)) = 0\,, \tag{33}$$

$$\left(\sum_{\{i_I\}}(A^{(-l+1)})^{\{m_I\}}_{\{i_I\}}\right)\psi(p^I(m_I)) + \ell\sum_J\left(\sum_{\{i_J\}}(i_J A^{(-l)})^{\{m_I\}}_{\{i_I\}}\right)\partial_J\psi(p^I(m_I)) = 0 \tag{34}$$

and so on.

Once again one has to appeal to recovery of the correct continuum equation (analogous to the classical limit) to select the leading order partial differential equation. In the case of general homogeneous loop quantum cosmology ($l = 2$ and no γ^{-1} term), this requires the differential equation to be second order.

For a non-zero solution, the bracket in eq.(33) must be zero. Using (rotational) symmetry conditions for the coefficients, we can reformulate this condition as

$$\sum_{i_I}(A^{(-l)})^{\{m_I\}}_{\{i_I\}} = 0 \tag{35}$$

(summing only over a single i_I) for all I and all choices of i_J for $J \neq I$.

The second equation (34) either gives us a first order differential equation or leads to further conditions on the coefficients of the difference equation and so on. *Let us assume for definiteness that the desired continuum equation is of second order.* Then the additional conditions for the coefficients are,

$$\sum_{\{i_I\}}(A^{(-l+1)})^{\{m_I\}}_{\{i_I\}} = 0 \tag{36}$$

and

$$\sum_{\{i_I\}}(i_J A^{(-l)})^{\{m_I\}}_{\{i_I\}} = \sum_{i_J} i_J \sum_{\{i_I\}_{I\neq J}}(A^{(-l)})^{\{m_I\}}_{\{i_I\}} = 0 \tag{37}$$

for all J (this cannot be split into several conditions by using symmetry).

For homogeneous loop quantum cosmology, the conditions on the coefficients are

$$\sum_{\{i_I\}}(A^{(-2)})^{\{m_I\}}_{\{i_I\}} \;=\; 0\,, \tag{38}$$

$$\sum_{\{i_I\}}(i_J A^{(-2)})^{\{m_I\}}_{\{i_I\}} \;=\; \sum_{i_J} i_J \sum_{\{i_I\}_{I\neq J\}}(A^{(-2)})^{\{m_I\}}_{\{i_I\}} \;=\; 0\,, \tag{39}$$

and are in fact satisfied for (10). In this case, the approximating partial differential equation is the Wheeler–DeWitt equation for the model.

The approximating *partial differential equation* becomes,

$$\ell^2 \sum_{J,K} \left(\sum_{\{i_I\}} i_J i_K (A^{(-l)})^{\{m_I\}}_{\{i_I\}} \right) \partial^2_{JK}\ \psi(p^I) + \ell \sum_J \left(\sum_{\{i_J\}} (i_J A^{(-l+1)})^{\{m_I\}}_{\{i_I\}} \right) \partial_J \psi(p^I(m_I)) + \left(\sum_{\{i_I\}} (A^{(-l+2)})^{\{m_I\}}_{\{i_I\}} \right) \psi(p^I) = 0\ . \tag{40}$$

Note that the bracket in the first term can be zero for some values of J, K but it has to be *non-zero* for those J, K which will reproduce the correct continuum equation. The desired continuum approximation thus gives us the condition (35, 36, 37).

B. Local Stability

The next step is to work with the difference equation in a local neighborhood and characterize the local *continuum* solutions (analogues of pre-classical solutions). These are Taylor expandable in γ and lead to a hierarchy of partial difference equations. The $s^{(0)}_{\{m_I\}}$ satisfies a partial difference equation with constant coefficients while the higher order terms in the Taylor expansion are determined recursively from *non-homogeneous* partial difference equations with constant coefficients. In a local neighborhood, we can take a short cut and directly take an ansatz for a local pre-classical solution as,

$$s_{\{(m_0)_I + \delta m_I\}} = a^{(0)}((m_0)_I) + \gamma \sum_I \delta m_I b^{(1)}_I((m_0)_I) + \gamma^2 \sum_{I,J} \delta m_I \delta m_J c^{(2)}_{IJ}((m_0)_I) + \cdots\ . \tag{41}$$

Substitution then implies that $a^{(0)}((m_0)_I), b^{(1)}_I((m_0)_I)$ are free parameters analogous to the case of ordinary difference equation. Now however there are additional possible free parameters, for example the $c^{(2)}_{IJ}((m_0)_I)$ for those I, J for which the first bracket in eqn(40) is zero. Similar argument implies that there could be free parameters at higher orders in the Taylor expansion (41). This is different from the case of ordinary difference equations where the higher order coefficients are all determined in terms of the $a^{(0)}, b^{(1)}$. The local continuum solution is thus parameterized by potentially infinitely many free parameters. Extension of a local continuum solution to the adjacent 'cells' of the given local neighborhood will be more complicated.

In the previous section, for formulation and analysis of 'local stability' we concentrated on the *locally approximated evolution*, equation (27). The analogue of this equation is the equation satisfied by $s^{(0)}_{\{m_I+i_I\}}$, namely,

$$\sum_{\{i_I\}} (A^{(-2)})_{\{i_I\}}\ s^{(0)}_{\{m_I+i_I\}} = 0\,. \tag{42}$$

The $(m_0)_I$ dependence of the coefficients is suppressed.

Unlike ordinary difference equations, this is not an evolution equation and we do not have a handle on its general solution. We can think of fixing $m_I + i_I, I \neq 1$ and allowing variation only along the 'first direction' and see if an evolution equation can be arrived at.

Let the lattice be D-dimensional. Let us introduce the notation:

$$\begin{aligned} Z^{i_2,i_3,\cdots,i_D}_{m_1}(m_2,m_3,\cdots,m_D) &:= s^{(0)}_{m_1,m_2+i_2,\cdots,m_D+i_D}\ , \\ X^{i_2,i_3,\cdots,i_D} &:= \sum_{i_1} (A^{(-2)})_{\{i_I\}}\ s^{(0)}_{\{m_I+i_I\}} \\ &= \sum_{i_1} (A^{(-2)})_{i_1,i_2,\cdots,i_D}\ Z^{i_2,i_3,\cdots,i_D}_{m_1+i_1}(m_2,m_3,\cdots,m_D) \end{aligned} \tag{43}$$

The equation (42) then can be expressed as,

$$\sum_{i_2,i_3,\cdots,i_D} X^{i_2,i_3,\cdots,i_D} \ = \ 0\,. \tag{44}$$

We can separate a particular set of values $i^*_2, i^*_3, \cdots, i^*_D$ and view the above equation as an equation for $X^{i^*_2,i^*_3,\cdots,i^*_D}$. Provided, $X^{i^*_2,i^*_3,\cdots,i^*_D}$, involves a non-trivial combination of $Z^{i^*_2,i^*_3,\cdots,i^*_D}_{m_1+i_1}(m_2,m_3,\cdots,m_D)$ (i.e. more than one m_1+i_1 occurring in the sum), we can think of the equation as an ordinary, non-homogeneous difference equation with respect to m_1, for the $Z^{i^*_2,i^*_3,\cdots,i^*_D}_{m_1+i_1}(m_2,m_3,\cdots,m_D)$. Thus we can interpret the partial difference equation as a (non-homogeneous) evolution equation for evolution along a line, parallel to the m_1 direction and passing through $m_2+i^*_2, m_3+i^*_3, \cdots, m_D+i^*_D$ in the lattice.

For example, in the context of homogeneous loop quantum cosmology, the lattice is three dimensional and the indices i_I range from -2 to $+2$. The locally approximated partial difference equation (42) turns out to be of the form,

$$\begin{aligned} & d(m_1)\left(s^{(0)}_{m_1,m_2+2,m_3+2} + s^{(0)}_{m_1,m_2-2,m_3-2} - s^{(0)}_{m_1,m_2+2,m_3-2} - s^{(0)}_{m_1,m_2-2,m_3+2}\right) \\ + & \ d(m_2)\left(s^{(0)}_{m_1+2,m_2,m_3+2} + s^{(0)}_{m_1-2,m_2,m_3-2} - s^{(0)}_{m_1-2,m_2,m_3+2} - s^{(0)}_{m_1+2,m_2,m_3-2}\right) \\ + & \ d(m_3)\left(s^{(0)}_{m_1+2,m_2+2,m_3} + s^{(0)}_{m_1-2,m_2-2,m_3} - s^{(0)}_{m_1-2,m_2+2,m_3} - s^{(0)}_{m_1+2,m_2-2,m_3}\right) \ = \ 0\ , \end{aligned} \tag{45}$$

where $d(m) \approx m^{-1}$ for large m.

The equation is invariant under permutation of the three directions, so it is enough to consider 'evolution' along the 'm_1'-direction. For the (i_2, i_3) set we have the eight possibilities $(2,2), (2,-2), (-2,2), (-2,-2), (2,0), (-2,0), (0,2), (0,-2)$ and the corresponding $Z^{i_2,i_3}_{m_1+i_1}$. The corresponding X^{i_2,i_3} involve non-trivial combinations of Z's only for the four sets $(2,0), (-2,0), (0,2), (0,-2)$. For example, $X^{2,0} = d(m_3)(Z^{2,0}_{m_1+2} - Z^{2,0}_{m_1-2})$ is a non-trivial combination while $X^{2,2} = d(m_1)Z^{2,2}_{m_1}$ is a trivial combination. Clearly, the partial difference equation can be viewed as an ordinary, non-homogeneous difference equation for any of the four non-trivial combinations. For each of these, the order of the evolution equation with respect to m_1 is 4. This illustrates how the locally approximated partial difference can be viewed as (different) non-homogeneous evolution equations for different combinations, X's.

Matching various local solutions across adjacent cells requires, in particular, matching along various directions connecting the adjacent cells. Now the arguments used for the case of ordinary difference equations can be invoked and issue of local stability becomes relevant. Note that these arguments are relevant *only* for the non-trivial combinations. For trivial combinations, local solutions in adjacent cells, across a cell boundary are *decoupled* and hence matching requires only equating the values on the cell boundary.

As observed before, the criteria for local stability depend only on the equation being an evolution equation and not on its (non-)homogeneity property. The criterion in terms of the characteristic roots of the associated *homogeneous* equation, $X^{i_2^*, i_3^*, \cdots, i_D^*} = 0$, then remains the same. This criterion is of course to be applied to *all* the evolution equations for different sets of $(i_2, i_3, \cdots, i_D)$ and along each of the directions. These details depend very much on the precise coefficients $A^{(-2)}_{i_1, i_2, \cdots, i_D}$ and are to be analyzed on a case by case basis.

For the equation of homogeneous loop quantum cosmology, eq.(45), the characteristic roots of the evolution equations are $\pm 1, \pm i$ which all have absolute value 1 thus satisfying the criterion of local stability [3].

To summarize: The rationale for continuum approximation and local stability remains the same as in the case of ordinary difference equation. In particular, with regards to local stability, one still deals with the locally approximated partial difference equation. Whenever this equation can be viewed as *an evolution* equation, we require this evolution to be locally stable as defined in the subsection III B. Since there could be multiple ways of viewing the partial difference equation as an evolution equation, local stability of all these evolutions is

required.

V. CONCLUSIONS

In this paper we have taken the view point that the fundamental equations of a theory arise from a discrete formulation and are thus difference equations. Typically such a fundamental discrete formulation arises from a quantization procedure which is not free of inherent ambiguities. A basic requirement on any such discrete formulation is that a continuum approximation should be available in suitable regimes. The parameter(s) controlling the discrete structure, being physical, cannot be taken to zero in arriving at a continuum approximation. A continuum approximation is then to be derived by invoking a suitable notion of pre-classicality without using unphysical, mathematical limits. In the context of isotropic loop quantum cosmology, we have shown how the definitions of slowly varying solutions together with γ-independence of the corresponding approximating differential equation suffices. Furthermore the formulation is powerful enough to put constraints on the coefficients of the fundamental equations as in equation (18). These are however necessary conditions for a continuum approximation to be admissible. The sufficiency can also be demonstrated in a well defined approximate sense by a constructive procedure provided, in addition, the condition of local stability (29) is satisfied.

More generally, the fundamental difference equation will be a *partial difference equation*. These arise for instance, in the context of the more general homogeneous models. The logic of admissibility of a continuum approximation is still valid and is seen to lead to similar conditions on the coefficients. One is then lead to a *locally approximated partial difference equation*. For the favorable case when the locally approximated partial difference equation can be viewed as a non-homogeneous, evolution equations along some direction for some $s^{(0)}_{\{m_I+i_I\}}$, one can extend local solutions across neighboring cells in a manner similar to the case of ordinary difference equation. The issue of stability of local approximations arises again for exactly the same reasons and leads to the definition of local stability exactly as before. The local stability criteria however are to be applied to *all* the evolution equations implied by the locally approximated partial difference equation. For the fundamental equation of the diagonal, homogeneous models, these evolution equations turn out to be stable [3].

When the fundamental difference equation is derived from a (quantum) theory, there are usually free parameters or other choices (e.g., quantization ambiguities). In such a case the conditions derived here can serve as powerful restriction criteria which go beyond the usual requirement that the quantization have the correct classical limit. Thus, they provide consistency checks for possible quantization approaches and restrict possible choices within a given approach. In this light, it is quite remarkable that the loop quantization of the homogeneous Bianchi I model, which is very close to the quantization of the full theory and does not allow many choices, turns out to satisfy our conditions automatically. This can be regarded as a positive consistency check for the full theory. For other homogeneous models, which have non-zero intrinsic curvature and are therefore not as close to the full theory, there are more choices which in particular allow violations of the conditions. In this case the conditions serve as strong selection criteria since it is possible to find quantizations fulfilling them for all those models [3].

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APPENDIX: AN EXAMPLE OF AN UNSTABLE QUANTIZATION

In this appendix we will focus on the isotropic subclass of the Bianchi-I and Bianchi-IX models and examine the stability properties of the evolution equations resulting from the quantization of the constraint operator as given originally in [9]. Again for our purposes, it is sufficient to consider only the vacuum case without the cosmological constant.

Let us recall some basic expressions: An orthonormal basis of states in the connection representation is given by $\langle c|n\rangle$ analogously to (7), which also forms an eigenbasis of the volume operator. In the isotropic case the convention $V_{\frac{|n|-1}{2}} = (\frac{1}{6}\gamma\ell_{\mathrm{P}}^2)^{\frac{3}{2}}\sqrt{(|n|-1)\ |n|\ (|n|+1)}$ has been used. As in the full theory [16], the constraint $\hat{H} = -\hat{H}_E + \hat{P}$ consists of two parts which can be quantized separately (in the main text we quantized the full constraint

directly which cannot be done in the full theory):

$$\hat{H}_E = 4i(\gamma\kappa\ell_{\rm P}^2)^{-1}\sum_{I,J,K}\epsilon^{IJK}\,{\rm tr}(h_I h_J h_I^{-1}h_J^{-1}h_{[I,J]}^{-1}h_K[h_K^{-1},\hat{V}])$$

where $h_{[I,J]} := \prod_K (h_K)^{C^K_{IJ}}$ with the structure constants $C^K_{IJ} = \epsilon_{IJK}$ for the closed isotropic model (Bianchi IX) and zero for the flat model (Bianchi I). The second part of the constraint operator is obtained from the first one by

$$\hat{P} = -8i(1+\gamma^{-2})\kappa^{-1}(\gamma\ell_{\rm P}^2)^{-3}\sum_{I,J,K}\epsilon^{IJK}\,{\rm tr}(h_I[h_I^{-1},\hat{K}]h_J[h_J^{-1},\hat{K}]h_K[h_K^{-1},\hat{V}])$$

with $\hat{K} = -\frac{i}{2}\gamma^{-2}\hbar^{-1}[\hat{H}_E,\hat{V}]\,\widehat{{\rm sgn}({\rm det}\,a_I^i)}$.

Upon transforming to the triad representation by $|s\rangle := \sum_{n\in\mathbb{Z}} S_n|n\rangle$, the constraint equation $\hat{H}|s\rangle = 0$ gets translated to a difference equation for the coefficients S_n which are just complex numbers in the vacuum case. The order of this equation is 16 for Bianchi-I and 20 for Bianchi-IX. Since the coefficient of S_0 is always zero in the factor ordering chosen for the constraint operator, one can ignore the sgn factors which enter through the definition of the $\hat{K}$ operator i.e. equivalently absorb them in the S_n's. Explicitly the equations are:

For Bianchi-I:

$$\begin{aligned}
0 \;=\; & -\Big(V_{\frac{|n+4|}{2}} - V_{\frac{|n+4|-2}{2}}\Big)S_{n+4} - \Big(V_{\frac{|n-4|}{2}} - V_{\frac{|n-4|-2}{2}}\Big)S_{n-4} \\
& +2\Big(V_{\frac{|n|}{2}} - V_{\frac{|n|-2}{2}}\Big)S_n \;+\; \Big(\frac{1+\gamma^{-2}}{4}\Big)\Big\{\;\Big(V_{\frac{|n+8|}{2}} - V_{\frac{|n+8|-2}{2}}\Big)\Big(k^+_{n+8}k^+_{n+4}\Big)S_{n+8} \\
& +\;\Big(V_{\frac{|n-8|}{2}} - V_{\frac{|n-8|-2}{2}}\Big)\Big(k^-_{n-8}k^-_{n-4}\Big)S_{n-8} - \frac{1}{2}\Big(V_{\frac{|n|}{2}} - V_{\frac{|n|-2}{2}}\Big)\Big(k^-_n k^+_{n+4} + k^+_n k^-_{n-4}\Big)S_n\Big\}
\end{aligned} \tag{A.1}$$

For Bianchi-IX:

$$\begin{aligned}
0 \;=\; & +\Big(V_{\frac{|n+5|}{2}} - V_{\frac{|n+5|-2}{2}}\Big)\Big(\frac{1-i}{2}\Big)S_{n+5} + \Big(V_{\frac{|n-5|}{2}} - V_{\frac{|n-5|-2}{2}}\Big)\Big(\frac{1+i}{2}\Big)S_{n-5} \\
& +\Big(V_{\frac{|n+3|}{2}} - V_{\frac{|n+3|-2}{2}}\Big)\Big(\frac{1+5i}{2}\Big)S_{n+3} + \Big(V_{\frac{|n-3|}{2}} - V_{\frac{|n-3|-2}{2}}\Big)\Big(\frac{1-5i}{2}\Big)S_{n-3} \\
& -\Big(V_{\frac{|n+1|}{2}} - V_{\frac{|n+1|-2}{2}}\Big)\Big(1+i\Big)S_{n+1} - \Big(V_{\frac{|n-1|}{2}} - V_{\frac{|n-1|-2}{2}}\Big)\Big(1-i\Big)S_{n-1} \\
& +\Big(\frac{1+\gamma^{-2}}{4}\Big)\sum_{k=-5}^{5}\Big(V_{\frac{|n-2k|}{2}} - V_{\frac{|n-2k|-2}{2}}\Big)\;A^{-2k}_{n-2k}\;S_{n-2k}
\end{aligned} \tag{A.2}$$

The definitions of $k^\pm_n$ and A^l_n are given in [9].

Observe that we must have infinitely many S_n to be non-zero for a non-trivial solution and the number of non-trivial independent solutions is reduced by one in both cases since

the coefficient of S_0 is zero (see [6]). We are interested in the admissibility of a continuum approximation and in particular in checking if the evolution is locally stable.

To explore these properties, we consider the large n behavior of the coefficients, S_n. For definiteness we take n to be positive and large which gets rid of the absolute values in the eigenvalues of the volume operator. Introduce the notation:

$$a^2(n) := \frac{\gamma \ell_{\rm P}^2 n}{6} \tag{A.3}$$

For large n one gets,

$$V_{\frac{n+b}{2}} \longrightarrow a^3\Big(1 + \frac{3}{2}\frac{b+1}{n} + \frac{3b^2+6b-1}{8n^2} + \cdots\Big) \tag{A.4}$$

$$k_n^{\pm} \longrightarrow 1 + o\Big(\frac{1}{n}\Big) \tag{A.5}$$

For Bianchi-IX one needs A_n^l defined in terms of $k_n^q := (\mathcal{K}_{n+1}^q - \mathcal{K}_{n-1}^q)/2$ for $q = \pm 1, \pm 3, \pm 5$. These become,

$$\mathcal{K}_{n+b}^{\pm 1} \longrightarrow \mp\Big(\frac{1 \mp i}{36}\Big)\Big\{ \mp 9n + \frac{9}{4} \mp 9b \ + \ o\Big(\frac{1}{n}\Big)\Big\} \tag{A.6}$$

$$\mathcal{K}_{n+b}^{\pm 3} \longrightarrow \pm\Big(\frac{5 \mp i}{36}\Big)\Big\{ \mp \frac{27}{2}n + \frac{81}{8} \mp \frac{27}{2}b \ + \ o\Big(\frac{1}{n}\Big)\Big\} \tag{A.7}$$

$$\mathcal{K}_{n+b}^{\pm 5} \longrightarrow \mp\Big(\frac{1 \pm i}{36}\Big)\Big\{ \mp \frac{45}{2}n + \frac{225}{8} \mp \frac{45}{2}b \ + \ o\Big(\frac{1}{n}\Big)\Big\} \tag{A.8}$$

$$k_{n+b}^{\pm 1} \longrightarrow +\Big(\frac{1 \mp i}{4}\Big) \ + \ o\Big(\frac{1}{n}\Big) \tag{A.9}$$

$$k_{n+b}^{\pm 3} \longrightarrow -\Big(\frac{3(5 \mp i)}{8}\Big) \ + \ o\Big(\frac{1}{n}\Big) \tag{A.10}$$

$$k_{n+b}^{\pm 5} \longrightarrow +\Big(\frac{5(1 \pm i)}{8}\Big) \ + \ o\Big(\frac{1}{n}\Big) \tag{A.11}$$

With these, the difference equations become:

$$0 \ = \ S_{n+4} - 2S_n + S_{n-4} - \frac{1+\gamma^{-2}}{4}(S_{n+8} - 2S_n + S_{n-8}) \qquad \text{(Bianchi-I)} \tag{A.12}$$

$$\begin{aligned} 0 \ = \ & (1-i)S_{n+5} + (1+i)S_{n-5} + (1+5i)S_{n+3} + (1-5i)S_{n-3} - 2(1+i)S_{n+1} \\ & -2(1-i)S_{n-1} + \Big(\frac{1+\gamma^{-2}}{2}\Big)\Big(\sum_{k=-5}^{5} A_{n-2k}^{-2k} S_{n-2k}\Big) \qquad \text{(Bianchi-IX)} \end{aligned} \tag{A.13}$$

The coefficients A_n^m satisfy the unitarity conditions $A_n^{-m} = (A_n^m)^*$ and are obtained as:

$$\begin{aligned} A_n^{10} \ &= \ +\frac{25}{32}i \\ A_n^{8} \ &= \ -\frac{45}{16} - \frac{15}{8}i \end{aligned}$$

$$
\begin{aligned}
A^6_n &= +4 - \frac{45}{32}i \\
A^4_n &= -\frac{3}{4} + \frac{7}{4}i \\
A^2_n &= -3 - \frac{59}{16}i \\
A^0_n &= +\frac{73}{8}
\end{aligned}
\tag{A.14}
$$

Note that in the large-n limit, the equation has reduced to a difference equation with constant coefficients and our first necessary condition for admissibility of continuum approximation is satisfied. The condition of Eq. (18) however is satisfied by the Bianchi-I case but *not* by the Bianchi-IX case. Therefore we will not get the Wheeler–DeWitt equation in the leading approximation for the latter.

A simple check on local stability is to obtain the characteristic roots to the *leading order in* γ. This picks out the polynomial obtained from the coefficients in the term multiplying γ^{-2}.

In both cases of Bianchi types, the corresponding characteristic polynomial is a polynomial in z^2. Hence both $\pm z$ are roots. The coefficients of the polynomial also come in complex conjugate pairs such that if z is a root then so is $1/z^*$. Furthermore, these polynomials are perfect squares, so each root is a *double* root. Corresponding to each such root, there will be solutions which will behave as $z^n(an + b)$.

For Bianchi-I, in the continuum approximation, the roots of the characteristic polynomial are the 8^{th} roots of unity each being a double root. Clearly all those roots have absolute values equal to 1 and exactly one root equals 1. Our condition of local stability is thus satisfied (see also [17] for the continuum limit).

For Bianchi-IX on the other hand, the roots are explicitly obtained as:

$$
\begin{aligned}
\lambda_0 &= 0.24077449441476 - i\ 0.47063359530671 \quad , \quad |\lambda_0| = 0.52864765032298 \\
\lambda_1 &= 0.29768070243356 + i\ 0.95466549083889 \quad , \quad |\lambda_1| = 1.0 \\
\lambda_2 &= 0.8 - i\ 0.6 \quad , \quad |\lambda_2| = 1.0 \\
\lambda_3 &= 0.86154480315169 - i\ 1.68403189553218 \quad , \quad |\lambda_3| = 1.89161911414729 \\
\lambda_4 &= -1.0 \quad , \quad |\lambda_4| = 1.0
\end{aligned}
\tag{A.15}
$$

The actual 10 double roots are $z_i^{\pm} = \pm\sqrt{\lambda_i}$. Observe that 1 is not a root of the characteristic polynomial. Hence the approximating differential equation equation will also have

lower derivative terms. Moreover we do have roots whose absolute values are not equal to one, thus violating local stability.

Thus, the original quantization in [9] for the closed isotropic model is not admissible by our criteria. In [12], a different quantization of the closed model has been derived with the methods of [3], providing an admissible quantization for this model.

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[18] All exact solutions of the fundamental equation have to satisfy the consistency condition following from the singularity avoidance mechanism [6, 13]. Given a local approximation (for $m \in [m_0, m_0 + \Delta m]$), we can evolve it backward by the exact fundamental difference equation. The consistency condition then will lead to a relation between $a^{(0)}(m_0), b^{(1)}(m_0)$ so that the solution is determined uniquely up to an over-all scaling (in the vacuum isotropic case). Such a solution is by construction, exact up to m_0 and approximates a locally pre-classical solution up to $m_0 + \Delta m$. The number of independent solutions is thus different from the Wheeler–DeWitt equation, but only after taking into account the behavior in the Planck regime.